Higher categorical symmetries of Lie groups

- Work in progress
- Physics applications joint w/ F. Burnell
- Banff Dec 2023

- G : finite group
- Sequence of n-categories

$$G_{(1)} \cong \mathbb{K}[G] = \text{group algebra} \cong \pi_{\leq 1}(BG)$$

$$G_{(2)} \cong \pi_{\leq 2}(BG) \cong \mathrm{Vec}_G$$

$$G_{(3)} \cong \pi_{\leq 3}(BG) \cong 2\text{-}\mathrm{Vec}_G$$

$$G_{(4)} \cong \dots\dots$$

$$\vdots$$

These higher categories underlay many important constructions in quantum topology and adjacent fields...

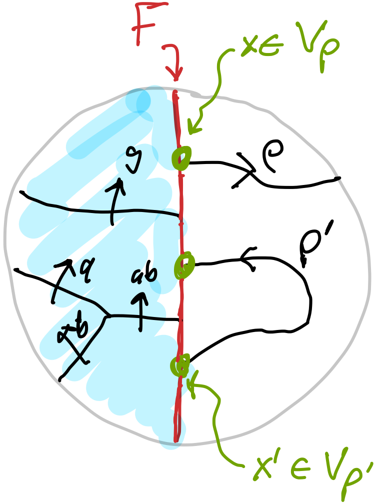
- colimit construction on $G_{(n)}$ → $n+1$-dim'l Dijkgraaf-Witten TQFTs
- G action on $(n-1)$-category ↔ module for $G_{(n)}$
- "non-invertible symmetries" ↔ completion of $G_{(n)}$
- G-graded ⊗-category ↔ $G_{(3)}$ acting on ⊗-cat.
- G-cross-braided category ↔ codim-2 defects in $G_{(3)}$
- G-SPT ↔ $G_{(n)}$ action on trivial $(n-1)$-category
- G-SET ↔ general $G_{(n)}$ action

Goal: generalize all of the above to G a compact Lie group

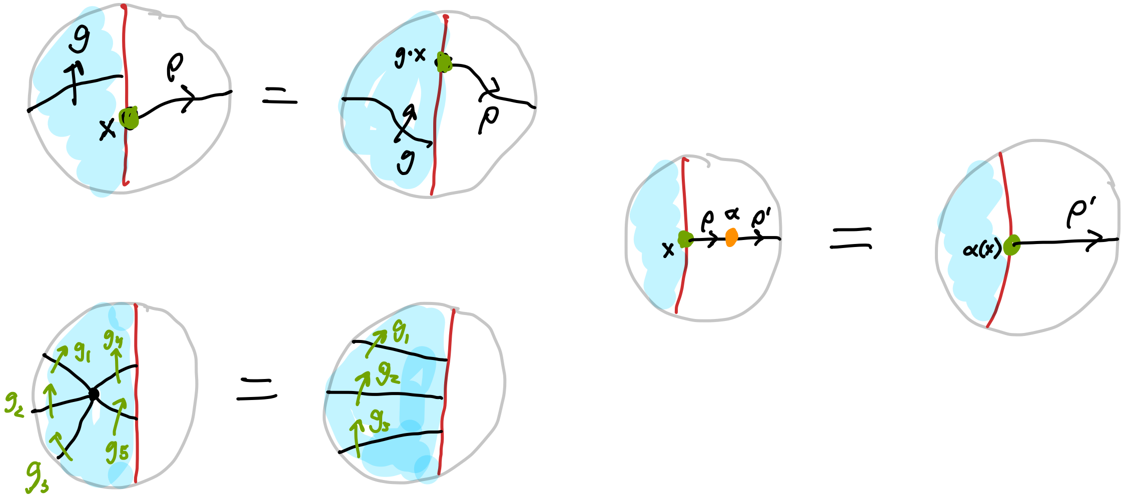
Recall Morita equivalence F between

$G_{(n)}$ and $\mathrm{Rep}(G)_{(n)}$ ← { symmetric monoidal category thought of as n-category

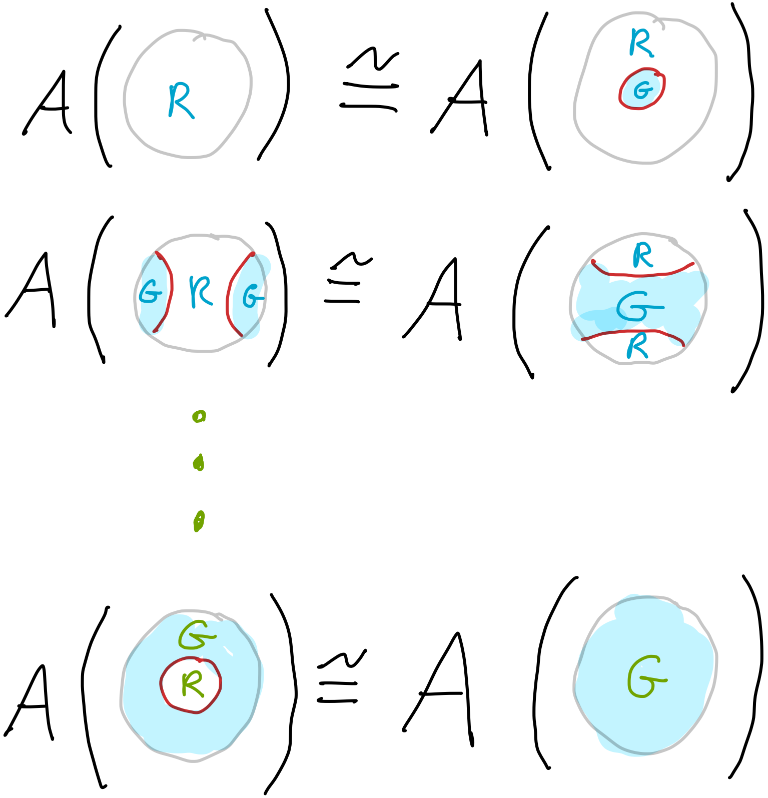
Generators:



Relations



Morita equivalence ↔ invertible domain wall



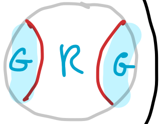
add G-0-handle

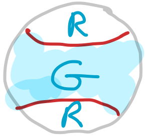
add G-1-handle

⋮

add G-n-handle

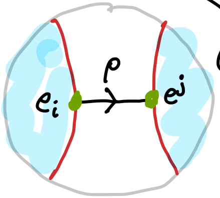
Only difficult case

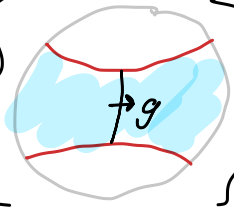
$A\Big($ 

$\Big) \cong A\Big($ 

$\Big)$ add G-1-handle ← difficult case

Bases:

$\Big\{$ 

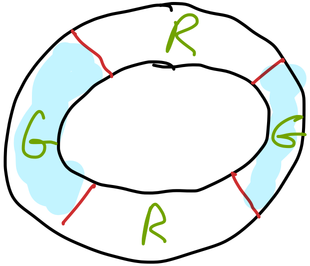
$\Big\}$ vs $\Big\{$ 

$\Big\}$

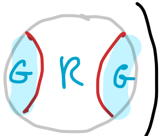
$\rho \in$ irreps $\qquad g \in G$

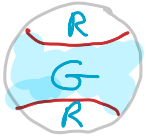
e_i, e^j basis

Action of



$S^{n-1} \times I$

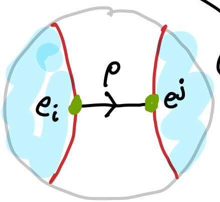
$A\Big($ 

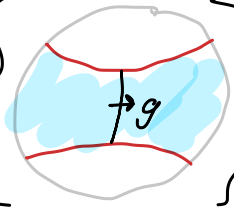
$\Big) \cong A\Big($ 

$\Big)$ add G-1-handle

difficult case

Bases:

$\Big\{$ 

$\Big\}$ vs $\Big\{$ 

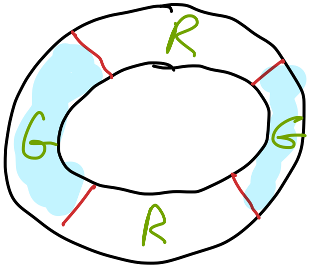
$\Big\}$

$\rho \in$ irreps

e_i, e^j basis

$g \in G$

Action of



$S^{n-1} \times I$

(Strong) Peter-Weyl Thm

$$\bigoplus_{\rho} \mathrm{End}_{\mathbb{C}}(\rho) \cong \mathbb{C}[G]$$

$\otimes$ of rep'ns $\longleftrightarrow$ pointwise multiplication

Composition of End $\longleftrightarrow$ G-convolution product

Idea: Let the various forms of the Peter-Weyl Thm for Lie groups dictate the definition of $G_{(n)}$. (i.e. preserve Morita equivalence.)

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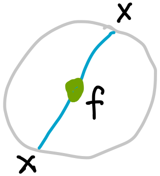
Peter-Weyl:
① algebraic functions on G ⟷ finite sums in ρ
② L^2 " " ⟷ ℓ^2 " "
③ C^∞ " " ⟷ rapidly decreasing ($n^k a_n$ bounded)
④ real analytic " " ⟷ exponentially decreasing ($K^n a_n$ bounded)

$\mathcal{F}(G) :=$ functions on G (focus on algebraic functions case)
$f : G \to \mathbb{C}$

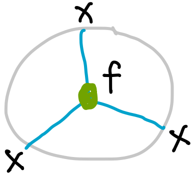
Defn of $G_{(2)}$ (G a Lie group)

If G is finite then $X \cong \bigoplus_{g\in G} g$

- one 0-morphism $*$
- one generating 1-morphism X
- 2-morphisms:

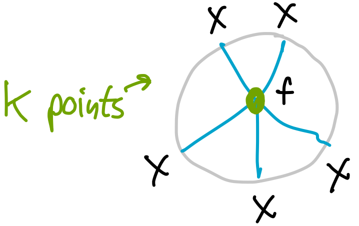


$$f \in \mathcal{F}(G) \quad \left(\mathrm{End}(x) \cong \mathcal{F}(G)\right)$$



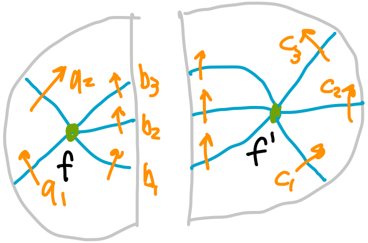
$$f \in \mathcal{F}\left(\{(g_1,g_2,g_3)\in G^3 \mid g_1g_2g_3=1\}\right)$$

$$\cong \mathcal{F}(G\times G) \cong \mathcal{F}(G)\otimes\mathcal{F}(G)$$



$$f\in\mathcal{F}\left(\{(g_i)\in G^k \mid \prod_i g_i = 1\}\right) \cong \mathcal{F}(G^{k-1})$$

• Composition of Σ-morphisms



$$(f \cdot f')(a_1, a_2, c_1, c_2, c_3) :=$$

$$\int f(a_1, a_2, b_1, b_2, b_3) \cdot f'(b_1, b_2, b_3, c_1, c_2, c_3,$$

$$\{(b_1, b_2, b_3) \mid b_1 b_2 b_3 = a_1 a_2 = c_1 c_2 c_3\}$$

$$\cong G \times G$$

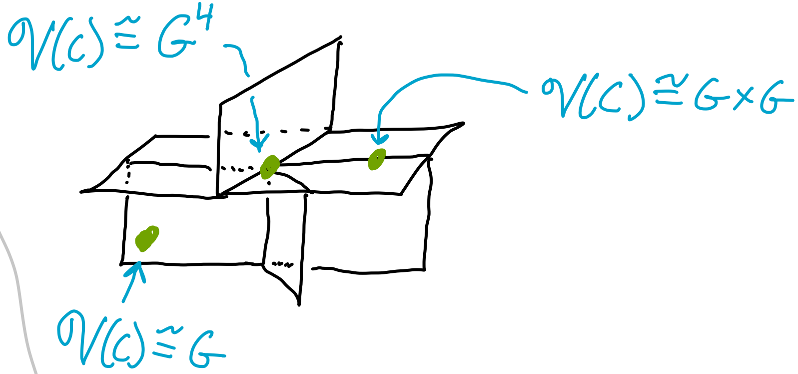
Definition of $G_{(n)}$

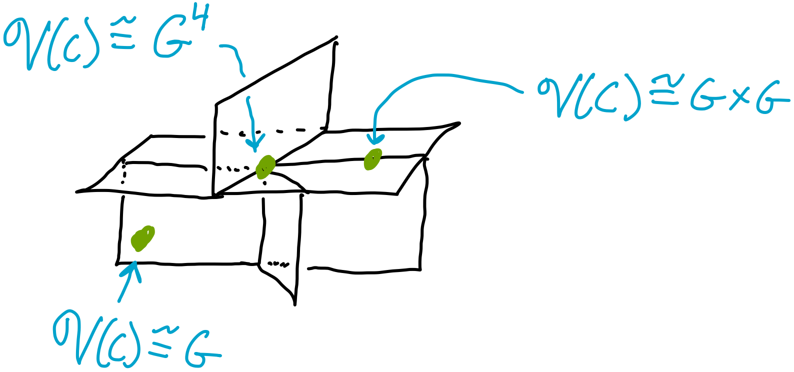
- 0-morphism: $*$
- 1-morphism: X

  $\left(X \cong \bigoplus_{g \in G} g \text{ if } G \text{ finite} \right)$
- $[2 \leq k \leq n-1]$-morphisms: one for each link type
- n-morphisms: $\mathcal{F}(\mathcal{V}(c))$

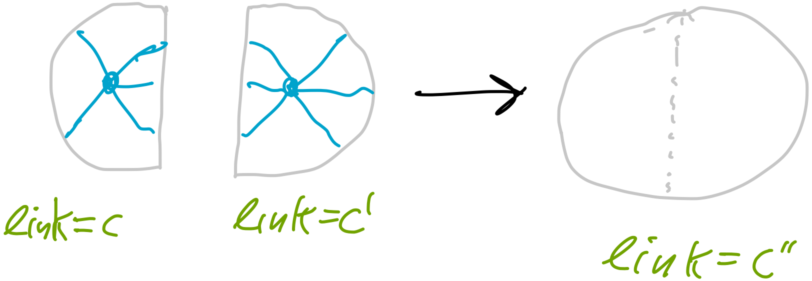
  c: link of vertex

$$\mathcal{V}(c) := \left\{ (g_i) \in G^{\text{codim-1 cells}} \;\middle|\; \forall \text{ codim-2 cells } e \quad \prod_{i=e} g_i = 1 \right\}$$





- composition of n-morphisms:



$$\begin{tikzcd}
\mathcal{V}(c)\times\mathcal{V}(c') \\
\mathcal{V}[\text{agree on common faces}] \arrow[u, hook] \arrow[d, "\text{proj}", "\text{fiber}\cong G^k"'] \\
\mathcal{V}(c'')
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(\mathcal{V}(c))\otimes\mathcal{F}(\mathcal{V}(c')) \arrow[d, "\cong"] \\
\mathcal{F}(\mathcal{V}(c)\times\mathcal{V}(c')) \arrow[d, "\text{restrict}"] \\
\mathcal{F}(\mathcal{V}[\cdots]) \arrow[d, "\text{integrate along fibers}"] \\
\mathcal{F}(\mathcal{V}(c''))
\end{tikzcd}$$

Note: $G_{(2)}$ is neither unital nor $\otimes$-unital, and similarly $G_{(n)}$ lacks units at all levels.

Nevertheless: Can make skein TQFT constructions work, using factorizability in place of $\otimes$-units

(All string diagrams must be "full", since we do not have the identity morphisms corresponding to empty space.)

TQFT based on $G_{(n)}$

$$A_{G_{(n)}}(M^n) := \mathbb{C}\left[G_{(n)} \text{ string diagrams on } M\right] \Big/ \sim$$

local relations

$$\cong \mathcal{F}\left[\{\pi_1(M) \to G\}\right]$$

$$\cong \mathcal{F}\left[\{\text{flat } G \text{ connections on } M\}\right]$$

$G_{(2)}$ acting on 1-cat

Recall G finite case:

$\boxed{0}$ $* \longmapsto$ 1-cat. C

$\boxed{1}$ $g \longmapsto$ functor $F_g : C \to C$
$(\forall g \in G)$

$\boxed{2}$ $(g,h) \longmapsto$ N.T. $N_{g,h} : F_g F_h \Rightarrow F_{g,h}$
$(\forall g,h \in G)$

such that

$$\begin{tikzcd}
 & F_{gh} F_k \arrow[dr, Rightarrow, "N_{gh,k}"] & \\
F_g F_h F_k \arrow[ur, Rightarrow, "N_{g,h} \bullet F_k"] \arrow[dr, Rightarrow, "F_g \bullet N_{h,k}"'] & \circlearrowleft & F_{ghk} \\
 & F_g F_{hk} \arrow[ur, Rightarrow, "N_{g,hk}"'] &
\end{tikzcd}$$

$G_{(2)}$ acting on 1-cat

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such that

$$\begin{tikzcd}
& F_{gh}F_k \arrow[dr, Rightarrow, "N_{gh,k}"] & \\
F_gF_hF_k \arrow[ur, Rightarrow, "N_{gh}\circ F_k"] \arrow[dr, Rightarrow, "F_g\circ N_{h,k}"'] & \circlearrowleft & F_{ghk} \\
& F_gF_{hk} \arrow[ur, Rightarrow, "N_{g,hk}"'] &
\end{tikzcd}$$

Example: $G \times Y \xrightarrow{\alpha} Y$

$C = Sh(Y)$

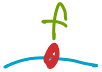
$F_x = Sh(Y) \xrightarrow{\pi^*} Sh(G \times Y) \xrightarrow{\alpha_*} Sh(Y)$

G : Lie gp

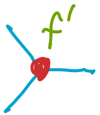
$\boxed{0}$ $* \longmapsto$ 1-cat C

$\boxed{1}$ $x \longmapsto F_x : C \to C$

$\boxed{2}$



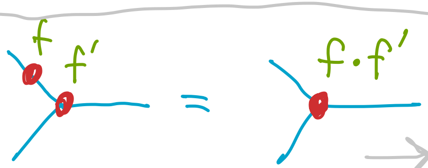
$f \in \mathcal{F}(G) \longmapsto N_f : F_x \Rightarrow F_x$



$f' \in \mathcal{F}(G \times G) \longmapsto N_{f'} : F_x F_x \Rightarrow F_x$

Such that .....

$$\begin{tikzcd}
& F_x \arrow[dr, Rightarrow, "N_{f_2}"] & \\
F_x \arrow[ur, Rightarrow, "N_{f_1}"] \arrow[rr, Rightarrow, "N_{f_1f_2}"'] & \circlearrowleft & F_x
\end{tikzcd}$$



$$\begin{tikzcd}
& F_xF_x \arrow[dr, Rightarrow, "N_{f'}"] & \\
F_xF_x \arrow[ur, Rightarrow, "N_f\circ F_x"] \arrow[rr, Rightarrow, "N_{ff'}"'] & \circlearrowleft & F_x
\end{tikzcd}$$

Another example: Line bundles and SPTs

$1 \to U(1) \to E \to G \to 1$ (classified by $H^2_{Borel}(G, U(1))$)

Associated line bundle
$$\begin{tikzcd} L \arrow[d] \\ G \end{tikzcd}$$

$\mathcal{F}(G)$ acts on $\mathrm{Sect}(L)$ $\rightsquigarrow$ module for $G_{(2)}$

Similarly, $H^K_{Borel}(G, U(1))$ gives module for $G_{(K)}$

(SPT for K-dimensional bulk)

$G_{(3)}$ acting on a $\otimes$-category

Recall finite G case:

$\boxed{0}$ $* \longmapsto$ $\otimes$-cat. C

$\boxed{1}$ $\forall g \in G$, $g \longmapsto$ $\otimes$-functor $F_g : C \to C$

$\boxed{2}$ $\forall g,h \in G$, $(g,h) \longmapsto$ pseudo-N.T. $N_{g,h} : F_g F_h \Rightarrow F_{gh}$

$\boxed{3}$ $\forall g,h,k \in G$ $(g,h,k) \longmapsto$

$$\begin{tikzcd}
 & F_{gh} F_k \arrow[dr, Rightarrow, "N_{gh,k}"] \arrow[dd, Rrightarrow, "M_{g,h,k}"] & \\
F_g F_h F_k \arrow[ur, Rightarrow, "N_{g,h} \cdot F_k"] \arrow[dr, Rightarrow, "F_g \cdot N_{h,k}"'] & & F_{ghk} \\
 & F_g F_{hk} \arrow[ur, Rightarrow, "N_{g,hk}"'] &
\end{tikzcd}$$

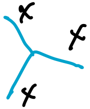
Satisfying, $\forall$ $g,h,k,l \in G$, pentagonish identity

$G_{(3)}$ acting on a ⊗-category, Lie group case

$\boxed{0}$ $* \mapsto$ ⊗-cat. $\mathcal{C}$

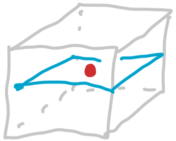
$\boxed{1}$ $x \mapsto$ ⊗-functor $F_x : \mathcal{C} \to \mathcal{C}$

$\boxed{2}$



$\mapsto$ pseudo N.T. $N_\lambda : F_x F_x \Rightarrow F_x$

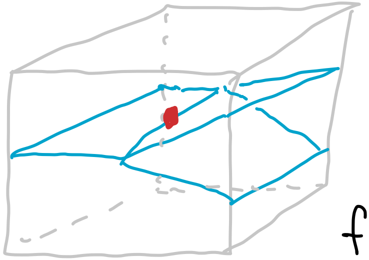
$\boxed{3}$



$f \in \mathcal{F}(G)$

$$\begin{tikzcd}
F_x \arrow[r, bend left, "id_{F_x}"] \arrow[r, bend right, "id_{F_x}"'] \arrow[r, phantom, "\Downarrow M_f"] & F_x
\end{tikzcd}$$

and more



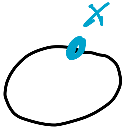
$f' \in \mathcal{F}(G \times G)$

$$\begin{tikzcd}
F_x F_x \arrow[r, bend left, "N_\lambda"] \arrow[r, bend right, "N_\lambda"'] \arrow[r, phantom, "\Downarrow M_{f'}"] & F_x
\end{tikzcd}$$

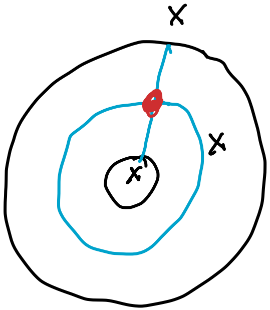
satisfying various relations

$G_{(2)}$: tube category / annular category / $G_{(2)}(S^1)$ / $\int_{S^1} G_{(2)}$

- one object



- $\mathrm{End}(\overset{x}{\bigcirc}) \cong \mathcal{F}(G\times G)$ (as vector space)



multiplication:

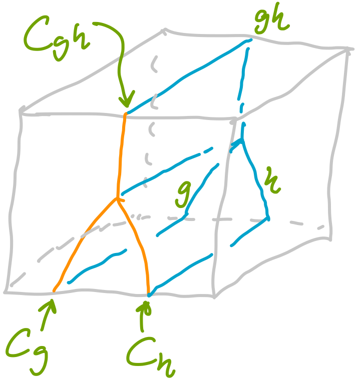
$$f\cdot f'(a,b)=\int_{g\in G} f(a,g)\cdot f'(g^{-1}ag,\, g^{-1}b)$$

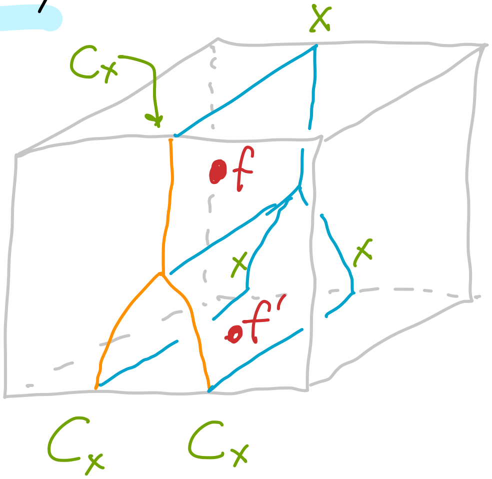
Note: $\mathrm{Rep}(G_{(2)}(S^1)) \cong$ G-equivariant sheaves for G acting on itself via conjugation

G-graded ⊗-category

G : Lie gp

G finite





$f, f' \in \mathcal{F}(G)$

- ⊗-cat C_X
- ∀ $f \in \mathcal{F}(G)$, N.T. $N_f : id_{C_X} \to id_{C_X}$

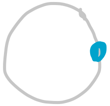
Satisfying $N_f(p \otimes q) = (N_{f'} p) \otimes (N_{f''} q)$

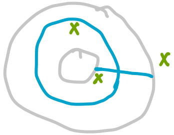
∀ $p, q \in C_X^0$ $f \in \mathcal{F}(G)$ $\Delta(f) = f' \otimes f''$ (Sweedler notation)

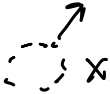
G-cross-braided categories

$$\text{G-cross-braided} \longleftrightarrow \text{module category for } \overset{\text{⊗-category}}{\overbrace{G_{(3)}(S^1)}}$$

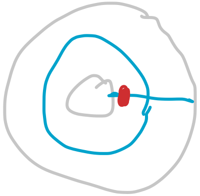
⇝ ⊗-prod + braiding

$G_{(3)}(S^1)$: $\boxed{0}$ – 

$\boxed{1}$ – 

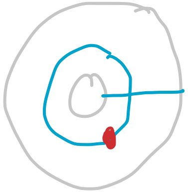
$\boxed{2}$ –  $\times I$, $f \in \mathcal{F}(G \times G)$

- vertical comp: pointwise on $G \times G$
- horizontal comp: $f \cdot f'(a,b) = \int_{g \in G} f(a,g) \cdot f'(g^{-1}ag, g^{-1}b)$

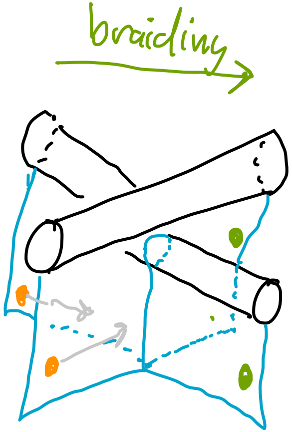


$\rightsquigarrow$ $N_f : id_C \to id_C$ (G-grading)

$f \in \mathcal{F}(G)$



$\rightsquigarrow$ G-action on C



$$G\times G \longrightarrow G\times G$$
$$(a,b) \longmapsto (aba^{-1}, a)$$
$$\mathcal{F}(G)\otimes\mathcal{F}(G) \longrightarrow \mathcal{F}(G)\otimes\mathcal{F}(G)$$

Cross braiding: $N_f \otimes N_f$ compatible w/